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LETTER TO THE EDITOR

Optical properties of the Compton effect

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Abstract. It is shown that the light reflection laws on a moving mirror are formally identical with the kinematical properties of the Compton effect.

In recent work, we showed how the Compton effect was characterised as a purely kinematical phenomenon involving the relativistic addition of velocities of the incident and scattered photons only (Elbaz 1987). In a previous paper, Ashworth and Jennison explained the Compton scattering as a pure specular reflection in the proper frame of reference of the scattered element (Ashworth and Jennison 1974).

In both cases, the relations involved contained only angles and velocities: angles of incidence and reflection, and velocities of light and of the recoil electron. The shift in frequency of the scattered x-ray coincides with the Doppler shift and is a consequence of the relativistic mechanical properties of a receding source (Ashworth and Jennison 1974). These features suggest that the Compton effect is strongly linked with the relativistic optical properties of matter and light.

In this letter, we propose to show that the laws of reflection of light on a moving mirror are formally identical with the kinematical properties of the Compton effect.

Such a result is not too surprising: since Hamilton, it is well known that a close correspondence exists between optical and mechanical equations.

In geometrical optics (Born and Wolf 1970, Marechal 1955), the light reflection law on a fixed mirror can be obtained by the expression of phase conservation

$$\omega t - \boldsymbol{k}_i \cdot \boldsymbol{r} = \omega t - \boldsymbol{k}_r \cdot \boldsymbol{r} \tag{1}$$

with

$$\omega^2 = k_i^2 c^2 = k_r^2 c^2. \tag{2}$$

If $\boldsymbol{u}_i$ and $\boldsymbol{u}_r$ are the unit vectors of the incident and reflected rays (Born and Wolf 1970) we can write, from (1) and (2),

$$ct - \boldsymbol{r} \cdot \boldsymbol{u}_i = ct - \boldsymbol{r} \cdot \boldsymbol{u}_r. \tag{3}$$

We set x and y coordinates in the plane of incidence so that $0x$ coincides with the normal of the mirror (figure 1). Relation (3) becomes

$$ct - x \cos \theta_i - y \sin \theta_i = ct + x \cos \theta_r - y \sin \theta_r. \tag{4}$$

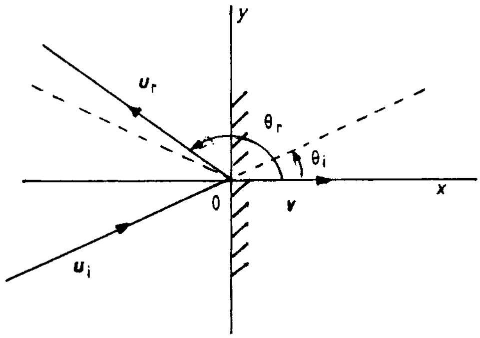

Figure 1. Reflection of light on a moving mirror.

By identification of the different terms we obtain

$$\cos\theta_i = -\cos\theta_r \tag{5}$$

$$\sin\theta_i = \sin\theta_r. \tag{6}$$

These equations are satisfied when

$$\theta_r = \pi - \theta_i. \tag{7}$$

This relation constitutes the well known law of reflection (Born and Wolf 1970).

When the mirror is moving uniformly with a velocity $v = \beta c$ along the direction $0x$ of its normal, the proper coordinates (x', t') are obtained from the laboratory coordinates (x, t) by the Lorentz transformation

$$\begin{cases} x' = \dfrac{x - vt}{(1-\beta^2)^{1/2}} \\ y' = y \\ t' = \dfrac{t - vx/c^2}{(1-\beta^2)^{1/2}} \end{cases} \qquad \begin{cases} x = \dfrac{x' + vt'}{(1-\beta^2)^{1/2}} \\ y = y' \\ t = \dfrac{t' + vx'/c^2}{(1-\beta^2)^{1/2}}. \end{cases} \tag{8}$$

The phase, which is a scalar, is a relativistic invariant (Landau and Lifshitz 1962, Furth 1970). From (1) we obtain

$$\omega_i t - \boldsymbol{k}_i \cdot \boldsymbol{r} = \omega_r t' - \boldsymbol{k}_r \cdot \boldsymbol{r}'. \tag{9}$$

In x, t and x', t' coordinates we obtain

$$\omega_i t - k_i x \cos\theta_i - k_i y \sin\theta_i = \omega_r t' + k_r x' \cos\theta_r - k_r y' \sin\theta_r. \tag{10}$$

By combining (10) and (8) we obtain

$$\begin{cases} \omega_r = \dfrac{\omega_i}{(1-\beta^2)^{1/2}}(1-\beta\cos\theta_i) \\ \cos\theta_r = \dfrac{-\cos\theta_i + \beta}{1-\beta\cos\theta_i} \\ \sin\theta_r = \dfrac{(1-\beta^2)^{1/2}\sin\theta_i}{1-\beta\cos\theta_i} \end{cases} \qquad \begin{cases} \omega_i = \dfrac{\omega_r}{(1-\beta^2)^{1/2}}(1-\beta\cos\theta_r) \\ \cos\theta_i = \dfrac{-\cos\theta_r + \beta}{1-\beta\cos\theta_r} \\ \sin\theta_i = \dfrac{(1-\beta^2)^{1/2}\sin\theta_r}{1-\beta\cos\theta_r}. \end{cases} \tag{11}$$

These relativistic equations generalise equations (2), (5) and (6), retrieved when $\beta = 0$, and constitute the light laws of reflection for a moving mirror.

They show that the incident and reflection angles play a symmetrical role: this is consistent with the classical light propagation laws, extended to the relativistic domain. This symmetry is more visible when we express the mirror velocity with respect to the angles θ_i and θ_r:

$$\beta = \frac{\cos\theta_i + \cos\theta_r}{1 + \cos\theta_i \cos\theta_r}. \tag{12}$$

If we call $v_i = c\cos\theta_i$ and $v_r = c\cos\theta_r$ the projections of the incident and reflected light velocities along the direction of the mirror motion, we obtain from (12)

$$v = \frac{v_i + v_r}{1 + v_i v_r / c^2}. \tag{13}$$

The mirror velocity is obtained by adding the projections relativistically of the incident and reflected light velocities along the direction of the motion.

The relations between the frequencies ω_i and ω_r of the incident and reflected light in (11) correspond to the Doppler effect (Landau and Lifshitz 1962).

The shift in orientation $\Delta\theta_r = \theta_r - (\theta_r)_0$ between the light rays reflected by the moving mirror and by the fixed one represents the 'angular aberration'. From (7) we find

$$\Delta\theta_r = \theta_r + \theta_i - \pi. \tag{14}$$

From (11) we calculate

$$\cos\Delta\theta_r = -\cos(\theta_i + \theta_r) = 1 - \frac{\beta^2 \sin^2\theta_i}{2(1 - \beta\cos\theta_i)}. \tag{15}$$

When $\Delta\theta_r$ is small, we can write

$$\cos\Delta\theta_r \simeq 1 - \tfrac{1}{2}(\Delta\theta_r)^2.$$

We find then, in first approximation in β,

$$\Delta\theta_r = \beta\sin\theta_i. \tag{16}$$

Figure 1 shows that the reflected ray diverges from the normal when the mirror is moving away from the incident ray along its normal.

This circumstance shows that the maximum angle of incidence θ_{iL}, or limit angle of incidence, is such that the corresponding reflected angle θ_{rL} is tangent to the mirror

$$\theta_{rL} = \tfrac{1}{2}\pi \Rightarrow \cos\theta_{iL} = \beta. \tag{17}$$

From (11) we find that the frequencies ω_{iL} of the incident and ω_{rL} of the scattered light rays are related by

$$\omega_{rL} = \omega_{iL}(1 - \beta^2)^{1/2}. \tag{18}$$

We now consider the Compton effect. Starting from the Compton equations (Compton 1923)

$$\begin{aligned} h\nu + m_0c^2 &= h\nu' + mc^2 \\ h\boldsymbol{\nu}/c &= h\boldsymbol{\nu}'/c + m\boldsymbol{v} \end{aligned} \tag{19}$$

with

$$m^2c^2 = m_0^2c^2 + m^2v^2 \tag{20}$$

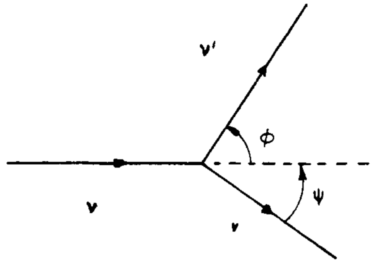

Figure 2. Collision between a photon and an electron.

we call ϕ the deviation angle of the scattered light with frequency ν' and ψ the deviation angle of the recoil electron (figure 2). The solution of the system (19) (Ashworth and Jennison 1974, Elbaz 1987) leads to the kinematical properties of the Compton effect:

$$\begin{cases} \nu' = \dfrac{\nu}{(1-\beta^2)^{1/2}}(1-\beta\cos\psi) \\ \cos(\phi+\psi) = \dfrac{-\cos\psi+\beta}{1-\beta\cos\psi} \end{cases} \qquad \begin{cases} \nu = \dfrac{\nu'}{(1-\beta^2)^{1/2}}[1-\beta\cos(\phi+\psi)] \\ \cos\psi = \dfrac{-\cos(\phi+\psi)+\beta}{1-\beta\cos(\phi+\psi)}. \end{cases} \tag{21}$$

When we compare figures 1 and 2, we notice that $\theta_r = \phi + \psi$ and $\theta_i = \psi$. We can then identify (21) and (11).

In the Compton effect, the kinematical properties of the collision between a photon and an electron at rest are formally identical with the laws of reflection of light on a mirror moving in the direction of its normal. As a consequence, the other properties of the light reflection on a moving mirror apply also to the Compton effect: symmetry between the incident and the scattered rays (Ashworth and Jennison 1974) and relativistic addition of velocities (Elbaz 1987).

The critical angle ψ_L between the incident ray and the recoil electron, such that $\cos\psi_L = \beta$, occurs when the scattered photon is normal to the electron $\boldsymbol{\nu}\cdot\boldsymbol{v}' = 0$ or $\phi + \psi = \frac{1}{2}\pi$. From (19) we then find

$$\nu' = \nu(1-\beta^2)^{1/2}. \tag{22}$$

This relation corresponds to relation (18) for a mirror. By combining (22) with (19) we obtain

$$h\nu = mc^2 \tag{23}$$

and

$$h\nu' = m_0c^2. \tag{24}$$

These results show that even if the Compton effect is interpreted in terms of particle collision, it is consistent with a light optical reflection description on a moving mirror.

References

Ashworth D G and Jennison R C 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 803-6

Born M and Wolf E 1970 *Principles of Optics* (Oxford: Pergamon) p 37

Compton A H 1923 *Phys. Rev.* **21** 715

Elbaz C 1987 *J. Phys. A: Math. Gen.* **20** 647-50

Furth R H 1970 *Fundamental Principles of Modern Theoretical Physics* (Oxford: Pergamon) p 192

Landau L D and Lifshitz E M 1962 *The Classical Theory of Fields* (Oxford: Pergamon) p 146

Marechal A 1955 *Handb. Phys.* **24** 16